

SURFACES WITH PARALLEL MEAN CURVATURE VECTOR IN COMPLEX SPACE FORMS

DOREL FETCU

ABSTRACT. We consider a quadratic form defined on the surfaces with parallel mean curvature vector of an any dimensional complex space form and prove that its $(2, 0)$ -part is holomorphic. When the complex dimension of the ambient space is equal to 2 we define a second quadratic form with the same property and then determine those surfaces with parallel mean curvature vector on which the $(2, 0)$ -parts of both of them vanish. We also provide a reduction of codimension theorem and prove a non-existence result for 2-spheres with parallel mean curvature vector.

1. INTRODUCTION

Almost sixty years ago H. Hopf was the first to use a quadratic form in order to study surfaces immersed in a 3-dimensional Euclidean space. He proved, in 1951, that any such surface which is homeomorphic to a sphere and has constant mean curvature is actually isometric to a round sphere (see [14]). This result was extended by S.-S. Chern to surfaces immersed in 3-dimensional space forms (see [8]) and by U. Abresch and H. Rosenberg to surfaces in simply connected, homogeneous 3-dimensional Riemannian manifolds, whose group of isometries has dimension 4 (see [1, 2]). Very recently, H. Alencar, M. do Carmo and R. Tribuzy have made the next step by obtaining Hopf-type results in spaces with dimension higher than 3, namely in product spaces $M^n(c) \times \mathbb{R}$, where $M^n(c)$ is a simply connected n -dimensional space form with constant sectional curvature $c \neq 0$ (see [3, 4]). They have considered the case of surfaces with parallel mean curvature vector, as a natural generalization of those with constant mean curvature in a 3-dimensional ambient space. We also have to mention a recent paper of F. Torralbo and F. Urbano, which is devoted to the study of surfaces with parallel mean curvature vector in $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{H}^2 \times \mathbb{H}^2$.

Minimal surfaces and surfaces with parallel mean curvature vector in complex space forms have been also a well studied subject in the last two decades (see, for example, [5, 7, 9, 10, 12, 15, 16, 17, 18]). In all these papers the Kähler angle proved to play a decisive role in understanding of the geometry of immersed surfaces in a complex space form, and, in several of them, important results were obtained when this angle was supposed to be constant (see [5, 16, 18]).

The main goal of our paper is to obtain some characterization results concerning surfaces with parallel mean curvature vector in complex space forms by using as a principal tool holomorphic quadratic forms defined on these surfaces. The paper is organized as follows. In Section 2 we introduce a quadratic form Q on surfaces of an arbitrary complex space form and prove that its $(2, 0)$ -part is holomorphic when the mean curvature vector of the surface is parallel. In Section 3 we work

2000 *Mathematics Subject Classification.* 53A10, 53C42, 53C55.

Key words and phrases. surfaces with parallel mean curvature vector, complex space forms, quadratic forms.

The author was supported by a Post-Doctoral Fellowship "Pós-Doutorado Júnior (PDJ)" offered by CNPq, Brazil.

in the complex space forms with complex dimension equal to 2 and find another quadratic form Q' with holomorphic $(2,0)$ -part. Then we determine surfaces with parallel mean curvature vector on which both $(2,0)$ -part of Q and $(2,0)$ -part of Q' vanish. As a by-product we reobtain a result in [12]. More precisely, we prove that a 2-sphere can be immersed as a surface with parallel mean curvature vector only in a flat complex space form and it is a round sphere in a hyperplane in \mathbb{C}^2 . In Section 4 we deal with surfaces in \mathbb{C}^n with parallel mean curvature vector, and we prove that the $(2,0)$ -part of Q vanishes on such a surface if and only if it is pseudo-umbilical. The main result of Section 5 is a reduction theorem, which states that a surface in a complex space form, with parallel mean curvature vector, either is totally real and pseudo-umbilical or it is not pseudo-umbilical and lies in a complex space form with complex dimension less or equal to 5. The last Section is devoted to the study of the 2-spheres with parallel mean curvature vector and constant Kähler angle. We prove that there are no non-pseudo-umbilical such spheres in a complex space form with constant holomorphic sectional curvature $\rho \neq 0$.

Acknowledgements. The author wants to thank Professor Harold Rosenberg for suggesting this subject, useful comments and discussions and constant encouragement.

2. A QUADRATIC FORM

Let Σ^2 be an immersed surface in $N^n(\rho)$, where N is a complex space form with complex dimension n , complex structure $(J, \langle \cdot, \cdot \rangle)$, and with constant holomorphic sectional curvature ρ ; which is $\mathbb{C}P^n(\rho)$, \mathbb{C}^n or $\mathbb{C}H^n(\rho)$, as $\rho > 0$, $\rho = 0$ and $\rho < 0$, respectively. Let us define a quadratic form Q on Σ^2 by

$$Q(X, Y) = 8|H|^2 \langle \sigma(X, Y), H \rangle + 3\rho \langle JX, H \rangle \langle JY, H \rangle,$$

where σ is the second fundamental form of the surface and H is its mean curvature vector field. Assume that H is parallel in the normal bundle of Σ^2 , i.e. $\nabla^\perp H = 0$, the normal connection ∇^\perp being defined by the equation of Weingarten

$$\nabla_X^N V = -A_V X + \nabla_X^\perp V,$$

for any vector field X tangent to Σ^2 and any vector field V normal to the surface, where ∇^N is the Levi-Civita connection on N and A is the shape operator.

We shall prove that the $(2,0)$ -part of Q is holomorphic. In order to do that, let us first consider the isothermal coordinates (u, v) on Σ^2 . Then $ds^2 = \lambda^2(du^2 + dv^2)$ and define $z = u + iv$, $\bar{z} = u - iv$, $dz = \frac{1}{\sqrt{2}}(du + idv)$, $d\bar{z} = \frac{1}{\sqrt{2}}(du - idv)$ and

$$Z = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial u} - i \frac{\partial}{\partial v} \right), \quad \bar{Z} = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right).$$

We also have $\langle Z, \bar{Z} \rangle = \langle \frac{\partial}{\partial u}, \frac{\partial}{\partial u} \rangle = \langle \frac{\partial}{\partial v}, \frac{\partial}{\partial v} \rangle = \lambda^2$.

In the following we shall calculate

$$\bar{Z}(Q(Z, Z)) = \bar{Z}(8|H|^2 \langle \sigma(Z, Z), H \rangle + 3\rho \langle JZ, H \rangle^2).$$

First, we get

$$\begin{aligned} \bar{Z}(\langle \sigma(Z, Z), H \rangle) &= \langle \nabla_{\bar{Z}}^N \sigma(Z, Z), H \rangle + \langle \sigma(Z, Z), \nabla_{\bar{Z}}^N H \rangle \\ &= \langle \nabla_{\bar{Z}}^\perp \sigma(Z, Z), H \rangle + \langle \sigma(Z, Z), \nabla_{\bar{Z}}^\perp H \rangle \\ &= \langle (\nabla_{\bar{Z}}^\perp \sigma)(Z, Z), H \rangle + \langle \sigma(Z, Z), \nabla_{\bar{Z}}^\perp H \rangle, \end{aligned}$$

where we have used that

$$(\nabla_{\bar{Z}}^\perp \sigma)(Z, Z) = \nabla_{\bar{Z}}^\perp \sigma(Z, Z) - 2\sigma(\nabla_{\bar{Z}} Z, Z) = \nabla_{\bar{Z}}^\perp \sigma(Z, Z)$$

since, from the definition of the connection ∇ on the surface, we easily get $\nabla_{\bar{Z}} Z = 0$.

Now, from the Codazzi equation, we obtain

$$\begin{aligned} (2.1) \quad \bar{Z}(\langle \sigma(Z, Z), H \rangle) &= \langle (\nabla_{\bar{Z}}^\perp \sigma)(\bar{Z}, Z), H \rangle + \langle (R^N(\bar{Z}, Z)Z)^\perp, H \rangle \\ &\quad + \langle \sigma(Z, Z), \nabla_{\bar{Z}}^\perp H \rangle \\ &= \langle (\nabla_{\bar{Z}}^\perp \sigma)(\bar{Z}, Z), H \rangle + \langle R^N(\bar{Z}, Z)Z, H \rangle + \langle \sigma(Z, Z), \nabla_{\bar{Z}}^\perp H \rangle. \end{aligned}$$

From the expression of the curvature tensor field of N

$$\begin{aligned} R^N(U, V)W &= \frac{\rho}{4} \{ \langle V, W \rangle U - \langle U, W \rangle V + \langle JV, W \rangle JU - \langle JU, W \rangle JV \\ &\quad + 2\langle JV, U \rangle JW \}, \end{aligned}$$

it follows

$$(2.2) \quad \langle R^N(\bar{Z}, Z)Z, H \rangle = \frac{3\rho}{4} \langle \bar{Z}, JZ \rangle \langle H, JZ \rangle.$$

We also have the following

Lemma 2.1.

$$(2.3) \quad \langle (\nabla_{\bar{Z}}^\perp \sigma)(\bar{Z}, Z), H \rangle = \langle \bar{Z}, Z \rangle \langle \nabla_{\bar{Z}}^\perp H, H \rangle.$$

Proof. By using the definition of $(\nabla_{\bar{Z}}^\perp \sigma)(\bar{Z}, Z)$ one obtains

$$(\nabla_{\bar{Z}}^\perp \sigma)(\bar{Z}, Z) = \nabla_{\bar{Z}}^\perp \sigma(\bar{Z}, Z) - \sigma(\nabla_{\bar{Z}} \bar{Z}, Z) - \sigma(\bar{Z}, \nabla_{\bar{Z}} Z) = \nabla_{\bar{Z}}^\perp \sigma(\bar{Z}, Z) - \sigma(\bar{Z}, \nabla_Z Z)$$

since $\nabla_Z \bar{Z} = 0$.

Next, let us consider the unit vector fields e_1 and e_2 corresponding to $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$, respectively, and $E = \frac{1}{\sqrt{2}}(e_1 - ie_2)$. Then we have $Z = \lambda E$ and

$$\sigma(\bar{Z}, Z) = \frac{\lambda^2}{2} \sigma(e_1 - ie_2, e_1 + ie_2) = \frac{\lambda^2}{2} (\sigma(e_1, e_1) + \sigma(e_2, e_2)) = \langle \bar{Z}, Z \rangle H.$$

Since $\nabla_Z Z$ is tangent it follows that $\nabla_Z Z = aZ + b\bar{Z}$ and then $0 = \langle \nabla_Z Z, Z \rangle = b\lambda^2$, where we have used the fact that $\langle Z, Z \rangle = 0$, and $a = \frac{1}{\lambda^2} \langle \nabla_Z Z, \bar{Z} \rangle$.

In conclusion

$$\begin{aligned} \langle (\nabla_{\bar{Z}}^\perp \sigma)(\bar{Z}, Z), H \rangle &= \langle \nabla_{\bar{Z}}^\perp (\langle \bar{Z}, Z \rangle H), H \rangle - \langle \nabla_Z Z, \bar{Z} \rangle \langle H, H \rangle \\ &= \langle \nabla_Z \bar{Z}, Z \rangle \langle H, H \rangle + \langle \nabla_Z Z, \bar{Z} \rangle \langle H, H \rangle \\ &\quad + \langle \bar{Z}, Z \rangle \langle \nabla_{\bar{Z}}^\perp H, H \rangle - \langle \nabla_Z Z, \bar{Z} \rangle \langle H, H \rangle \\ &= \langle \bar{Z}, Z \rangle \langle \nabla_{\bar{Z}}^\perp H, H \rangle. \end{aligned}$$

□

Lemma 2.2.

$$(2.4) \quad \bar{Z}(\langle JZ, H \rangle^2) = 2\langle JZ, H \rangle \langle (JZ)^\perp, \nabla_{\bar{Z}}^\perp H \rangle - 2|H|^2 \langle \bar{Z}, JZ \rangle \langle JZ, H \rangle$$

Proof. From the definitions of the Kähler structure and of the Levi-Civita connection we have

$$\begin{aligned}
\bar{Z}(\langle JZ, H \rangle^2) &= 2\langle JZ, H \rangle \{ \langle \nabla_{\bar{Z}}^N JZ, H \rangle + \langle JZ, \nabla_{\bar{Z}}^N H \rangle \} \\
&= 2\langle JZ, H \rangle \{ \langle \bar{Z}, Z \rangle \langle JH, H \rangle - \langle (JZ)^\top, A_H \bar{Z} \rangle \\
&\quad + \langle (JZ)^\perp, \nabla_{\bar{Z}}^\perp H \rangle \} \\
&= 2\langle JZ, H \rangle \{ \langle (JZ)^\perp, \nabla_{\bar{Z}}^\perp H \rangle - \langle \sigma((JZ)^\top, \bar{Z}), H \rangle \} \\
&= 2\langle JZ, H \rangle \{ \langle (JZ)^\perp, \nabla_{\bar{Z}}^\perp H \rangle - \langle JZ, \bar{Z} \rangle |H|^2 \},
\end{aligned}$$

where we have used $\nabla_{\bar{Z}}^N Z = \sigma(\bar{Z}, Z) = \langle \bar{Z}, Z \rangle H$, as we have seen in the proof of the previous Lemma, and $(JZ)^\top = \frac{1}{\lambda^2} \langle JZ, \bar{Z} \rangle Z$, that can be easily checked. \square

By replacing (2.2), (2.3) and (2.4) into (2.1) we obtain that $\bar{Z}(Q(Z, Z))$ vanishes and then we come to the conclusion that

Proposition 2.3. *If Σ^2 is an immersed surface in a complex space form $N^n(\rho)$, with parallel mean curvature vector field, then the $(2, 0)$ -part of the quadratic form Q , defined on Σ^2 by*

$$Q(X, Y) = 8|H|^2 \langle \sigma(X, Y), H \rangle + 3\rho \langle JX, H \rangle \langle JY, H \rangle,$$

is holomorphic.

3. QUADRATIC FORMS AND 2-SPHERES IN 2-DIMENSIONAL COMPLEX SPACE FORMS

In this section we shall define a new quadratic form on a surface Σ^2 immersed in a complex space form $N^2(\rho)$, with parallel mean curvature vector field $H \neq 0$, and prove that its $(2, 0)$ -part is holomorphic. Then, by using these two quadratic forms, we shall classify the 2-spheres with nonzero parallel mean curvature vector.

3.1. Another quadratic form. Let us consider an oriented orthonormal local frame $\{\tilde{e}_1, \tilde{e}_2\}$ on the surface and denote by θ the Kähler angle function defined by

$$\langle J\tilde{e}_1, \tilde{e}_2 \rangle = \cos \theta.$$

The immersion $x : \Sigma^2 \rightarrow N$ is said to be holomorphic if $\cos \theta = 1$, anti-holomorphic if $\cos \theta = -1$, and totally real if $\cos \theta = 0$. In the following we shall assume that x is neither holomorphic or anti-holomorphic.

Next, we take $e_3 = -\frac{H}{|H|}$ and let e_4 be the unique unit normal vector field orthogonal to e_3 compatible with the orientation of Σ^2 in N . Since e_3 is parallel in the normal bundle so is e_4 , and, as the Kähler angle is independent of the choice of the orthonormal frame on the surface (see, for example, [9]), we have

$$(3.1) \quad \langle Je_4, e_3 \rangle = \cos \theta.$$

Now, we can consider the vector fields

$$e_1 = \cot \theta e_3 - \frac{1}{\sin \theta} J e_4, \quad e_2 = \frac{1}{\sin \theta} J e_3 + \cot \theta e_4$$

tangent to the surface and obtain an orthonormal frame field $\{e_1, e_2, e_3, e_4\}$ adapted to Σ^2 in N .

We define a quadratic form Q' on Σ^2 by

$$Q'(X, Y) = 8i|H|\langle\sigma(X, Y), e_4\rangle + 3\rho\langle JX, e_4\rangle\langle JY, e_4\rangle$$

and again consider the isothermal coordinates (u, v) on Σ^2 and the tangent complex vector fields Z and \bar{Z} . In the same way as in the case of Q , using the Codazzi equation, the fact that H and e_4 are parallel and the expression of the curvature vector field of N , we get

$$(3.2) \quad \bar{Z}(\langle\sigma(Z, Z), e_4\rangle) = \frac{3\rho}{4}\langle\bar{Z}, JZ\rangle\langle JZ, e_4\rangle.$$

On the other hand, we have

$$\begin{aligned} \bar{Z}(\langle JZ, e_4\rangle^2) &= 2\langle JZ, e_4\rangle\{\langle\nabla_{\bar{Z}}^N JZ, e_4\rangle + \langle JZ, \nabla_{\bar{Z}}^N e_4\rangle\} \\ &= 2\langle JZ, e_4\rangle\{\langle\bar{Z}, Z\rangle\langle JH, e_4\rangle - \langle(JZ)^\top, A_{e_4}\bar{Z}\rangle\} \\ &= -2|H|\langle JZ, e_4\rangle\langle\bar{Z}, Z\rangle\langle Je_3, e_4\rangle - 2\langle JZ, e_4\rangle\langle\sigma((JZ)^\top, \bar{Z}), e_4\rangle \\ &= 2|H|\langle JZ, e_4\rangle\langle\bar{Z}, Z\rangle\cos\theta - 2\langle JZ, e_4\rangle\langle JZ, \bar{Z}\rangle\langle H, e_4\rangle \\ &= 2|H|\langle JZ, e_4\rangle\langle\bar{Z}, Z\rangle\cos\theta, \end{aligned}$$

where we have used $\nabla_{\bar{Z}}^N Z = \sigma(\bar{Z}, Z) = \langle\bar{Z}, Z\rangle H$, $(JZ)^\top = \frac{1}{\lambda^2}\langle JZ, \bar{Z}\rangle Z$ and (3.1). But $\langle\bar{Z}, JZ\rangle = -i\langle\bar{Z}, Z\rangle\langle e_1, Je_2\rangle = i\langle\bar{Z}, Z\rangle\cos\theta$, and therefore

$$(3.3) \quad \bar{Z}(\langle JZ, e_4\rangle^2) = -2i|H|\langle\bar{Z}, JZ\rangle\langle JZ, e_4\rangle.$$

Hence, from (3.2) and (3.3), one obtains $\bar{Z}(Q'(Z, Z)) = 0$, which means that the $(2, 0)$ -part of the quadratic form Q' is holomorphic.

3.2. 2-Spheres in 2-dimensional complex space forms. In order to classify the 2-spheres in 2-dimensional complex space forms, we shall need a result of T. Ogata in [16], which we will briefly recall in the following (see also [12] and [15]). Consider a surface Σ^2 isometrically immersed in a complex space form $N^2(\rho)$, with parallel mean curvature vector field $H \neq 0$. Using the frame field on $N^2(\rho)$ adapted to Σ^2 , defined above, and considering isothermal coordinates (u, v) on the surface, Ogata proved that there exist complex-valued functions a and c on Σ^2 such that θ , λ , a and c satisfy

$$(3.4) \quad \begin{cases} \frac{\partial\theta}{\partial z} = \lambda(a + b) \\ \frac{\partial\lambda}{\partial \bar{z}} = -|\lambda|^2(\bar{a} - b)\cot\theta \\ \frac{\partial a}{\partial \bar{z}} = \bar{\lambda}\left(2|a|^2 - 2ab + \frac{3\rho\sin^2\theta}{8}\right)\cot\theta \\ \frac{\partial c}{\partial z} = 2\lambda(a - b)c\cot\theta \\ |c|^2 = |a|^2 + \frac{\rho(3\sin^2\theta - 2)}{8} \end{cases}$$

where $z = u + iv$ and $|H| = 2b$; and also the converse: if ρ is a real constant, b a positive constant, Σ^2 a 2-dimensional Riemannian manifold, and there exist some functions θ , a and c on Σ^2 satisfying (3.4), then there is an isometric immersion of Σ^2 into $N^2(\rho)$ with parallel mean curvature vector field of length equal to $2b$ and with the Kähler angle θ . The second fundamental form of Σ^2 in N w.r.t. $\{e_1, e_2, e_3, e_4\}$ is given by

$$\sigma^3 = \begin{pmatrix} -2b - \Re(\bar{a} + c) & -\Im(\bar{a} + c) \\ -\Im(\bar{a} + c) & -2b + \Re(\bar{a} + c) \end{pmatrix} \quad \text{and} \quad \sigma^4 = \begin{pmatrix} \Im(\bar{a} - c) & -\Re(\bar{a} - c) \\ -\Re(\bar{a} - c) & -\Im(\bar{a} - c) \end{pmatrix}$$

and the Gaussian curvature of Σ^2 is $K = 4b^2 - 4|c|^2 + \frac{\rho}{2}$ (see also [12]).

Assume now that the $(2, 0)$ -part of Q and the $(2, 0)$ -part of Q' vanish on the surface Σ^2 . It follows, from the expression of the second fundamental form, that $\bar{c} + a \in \mathbb{R}$, $\bar{c} - a \in \mathbb{R}$ and

$$32b(\bar{c} + a) - 3\rho \sin^2 \theta = 0, \quad 32b(\bar{c} - a) + 3\rho \sin^2 \theta = 0.$$

Therefore $c = 0$ and $a = \frac{3\rho \sin^2 \theta}{32b}$ and, from the fifth equation of (3.4), it follows

$$(3.5) \quad 9\rho^2 \sin^4 \theta + 128\rho b^2(3 \sin^2 \theta - 2) = 0.$$

We have to split the study of this equation in two cases. First, if $\rho = 0$ then the above equation holds and $a = 0$. Next, if $\rho \neq 0$, we get that function θ is a constant. This, together with the first equation of (3.4), lead to $a = \frac{3\rho \sin^2 \theta}{32b} = -b$. By replacing in equation (3.5) we obtain $\rho = -12b^2$ and then $\sin^2 \theta = \frac{8}{9}$. We note that in both cases the Gaussian curvature of Σ^2 is given by $K = 4b^2 + \frac{\rho}{2} = \text{constant}$ (see [12]). Thus, by using Theorem 1.1 in [12], we have just proved that

Theorem 3.1. *If the $(2, 0)$ -part of Q and the $(2, 0)$ -part of Q' vanish on a surface Σ^2 isometrically immersed in a complex space form $N^2(\rho)$, with parallel mean curvature vector field of length $2b > 0$, then either*

- (1) $N^2(\rho) = \mathbb{C}H^2(-12b^2)$ and Σ^2 is the slant surface in [6] (Theorem 3(2));
- (2) $N^2(\rho) = \mathbb{C}^2$ and Σ^2 is a part of a round sphere in a hyperplane in \mathbb{C}^2 .

Since the Gaussian curvature K is nonnegative only in the second case of the Theorem, we have also reobtained the following result of S. Hirakawa in [12].

Corollary 3.2. *If \mathbb{S}^2 is an isometrically immersed sphere in a 2-dimensional complex space form, with nonzero parallel mean curvature vector, then it is a round sphere in a hyperplane in \mathbb{C}^2 .*

4. A REMARK ON THE 2-SPHERES IN \mathbb{C}^n

Proposition 4.1. *Let Σ^2 be an isometrically immersed surface in \mathbb{C}^n , with nonzero parallel mean curvature vector. Then the $(2, 0)$ -part of the quadratic form Q vanishes on Σ^2 if and only if the surface is pseudo-umbilical, i.e. $A_H = |H|^2 I$.*

Proof. It can be easily seen that if Σ^2 is pseudo-umbilical then the $(2, 0)$ -part of Q vanishes and, therefore, we have to prove only the necessity.

From $Q(Z, Z) = \frac{\langle Z, \bar{Z} \rangle^2}{2} Q(e_1 - ie_2, e_1 - ie_2) = 0$ it follows

$$\langle \sigma(e_1, e_1) - \sigma(e_2, e_2), H \rangle = 0$$

and

$$\langle \sigma(e_1, e_2), H \rangle = 0.$$

But, since $\langle \sigma(e_1, e_1) + \sigma(e_2, e_2), H \rangle = 2|H|^2$, we obtain, for each $i \in \{1, 2\}$,

$$\langle A_H e_i, e_i \rangle = \langle \sigma(e_i, e_i), H \rangle = |H|^2.$$

Therefore $A_H = |H|^2 I$, i.e. Σ^2 is pseudo-umbilical. \square

S.-T. Yau proved (Theorem 4 in [21]) that if Σ^2 is a surface with parallel mean curvature vector H in a manifold N with constant sectional curvature, then either Σ^2 is a minimal surface of an umbilical hypersurface of N or Σ^2 lies in a 3-dimensional umbilical submanifold of N with constant mean curvature, as H is an umbilical direction or the second fundamental form of Σ^2 can be diagonalized simultaneously. We note that, in the first case, the mean curvature vector field of Σ^2 in \mathbb{C}^n is orthogonal to the hypersurface.

Applying this result, together with Proposition 4.1, to the 2-spheres in \mathbb{C}^n , and using the Gauss equation of a hypersurface in \mathbb{C}^n , we get

Proposition 4.2. *If S^2 is an isometrically immersed sphere in \mathbb{C}^n , with nonzero parallel mean curvature vector field H , then it is a minimal surface of a hypersphere $S^{2n-1}(|H|) \subset \mathbb{C}^n$.*

5. REDUCTION OF THE CODIMENSION

Let $x : \Sigma^2 \rightarrow N^n(\rho)$, $n \geq 3$, $\rho \neq 0$, be an isometric immersion of a surface Σ^2 in a complex space form, with parallel mean curvature vector field $H \neq 0$.

Lemma 5.1. *For any vector V normal to Σ^2 , which is also orthogonal to $JT\Sigma^2$ and to JH , we have $[A_H, A_V] = 0$, i.e. A_H commutes with A_V .*

Proof. The statement follows easily, from the Ricci equation

$$\langle R^\perp(X, Y)H, V \rangle = \langle [A_H, A_V]X, Y \rangle + \langle R^N(X, Y)H, V \rangle,$$

since

$$\begin{aligned} \langle R^N(X, Y)H, V \rangle &= \frac{\rho}{4} \{ \langle JY, H \rangle \langle JX, V \rangle - \langle JX, H \rangle \langle JY, V \rangle \\ &\quad + 2 \langle JY, X \rangle \langle JH, V \rangle \} \\ &= 0 \end{aligned}$$

and $R^\perp(X, Y)H = 0$. □

Remark 5.2. If $n = 3$ and $H \perp JT\Sigma^2$ do not hold simultaneously, then there exists at least one normal vector V as in Lemma 5.1. This can be proved by using the basis of the tangent space TN along Σ^2 defined in [17], which construction we shall briefly explain in the following. Let us consider a local orthonormal frame $\{e_1, e_2\}$ of vector fields tangent to Σ^2 . Since we have assumed that $H \neq 0$ it follows that Σ^2 is not holomorphic or antiholomorphic, which means that $\cos^2 \theta = 1$ only at isolated points, and we shall work in the open dense set of points where $\cos^2 \theta \neq 1$, where θ is the Kähler angle function. The next step is to define two normal vectors by

$$e_3 = -\cot \theta e_1 - \frac{1}{\sin \theta} J e_2 \quad \text{and} \quad e_4 = \frac{1}{\sin \theta} J e_1 - \cot \theta e_2$$

and now we have an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of $\text{span}\{e_1, e_2, J e_1, J e_2\}$. Moreover, we can set

$$\begin{aligned} \tilde{e}_1 &= \cos\left(\frac{\theta}{2}\right) e_1 + \sin\left(\frac{\theta}{2}\right) e_3, & \tilde{e}_2 &= \cos\left(\frac{\theta}{2}\right) e_2 + \sin\left(\frac{\theta}{2}\right) e_4 \\ \tilde{e}_3 &= \sin\left(\frac{\theta}{2}\right) e_1 - \cos\left(\frac{\theta}{2}\right) e_3, & \tilde{e}_4 &= -\sin\left(\frac{\theta}{2}\right) e_2 + \cos\left(\frac{\theta}{2}\right) e_4 \end{aligned}$$

and obtain a J -canonical basis of $\text{span}\{e_1, e_2, J e_1, J e_2\}$, i.e. $J \tilde{e}_{2i-1} = \tilde{e}_{2i}$. Finally, let us consider a J -basis of TN along Σ^2 , of the form $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4, \tilde{e}_5, \tilde{e}_6 = J \tilde{e}_5, \dots, \tilde{e}_{2n-1}, \tilde{e}_{2n} = J \tilde{e}_{2n-1}\}$. Now, three situations can occur:

- (1) $H \in (JT\Sigma^2)^\perp$, and then $\tilde{e}_5 \perp JT\Sigma^2$ and $\tilde{e}_5 \perp JH$, where we have denoted by $(JT\Sigma^2)^\perp = \{(JX)^\perp : X \text{ tangent to } \Sigma^2\}$;
- (2) $H \perp JT\Sigma^2$, and then, if we choose $\tilde{e}_5 = H$ and $\tilde{e}_6 = JH$, we have $\tilde{e}_7 \perp JT\Sigma^2$ and $\tilde{e}_7 \perp JH$ (obviously, this case can occur only if $n > 3$);
- (3) $H \notin (JT\Sigma^2)^\perp$ and H is not orthogonal to $JT\Sigma^2$. In this case we may consider the vector u , the projection of H on the complementary space of $(JT\Sigma^2)^\perp$ in TN (along Σ^2) and set $\tilde{e}_5 = \frac{u}{|u|}$. It follows that $\tilde{e}_5 \perp JT\Sigma^2$ and $\tilde{e}_5 \perp JH$.

If $n = 3$ and $H \perp JT\Sigma^2$ it is easy to see that

$$\langle R^N(X, Y)H, e_3 \rangle = \langle R^N(X, Y)H, e_4 \rangle = 0$$

for any vector fields X and Y tangent to Σ^2 , and then that A_H commutes with A_{e_3} and A_{e_4} .

Conclusively, we get the following

Corollary 5.3. *Either H is an umbilical direction or there exists a basis that diagonalizes simultaneously A_H and A_V , for all normal vectors satisfying $V \perp JH$, if $n = 3$ and $H \perp JT\Sigma^2$, or the conditions in Lemma 5.1, otherwise.*

Lemma 5.4. *Assume that H is nowhere an umbilical direction. Then there exists a parallel subbundle of the normal bundle which contains the image of the second fundamental form σ and has dimension less or equal to 8.*

Proof. We consider the following subbundle L of the normal bundle

$$L = \text{span}\{\text{Im } \sigma \cup (J \text{Im } \sigma)^\perp \cup (JT\Sigma^2)^\perp\},$$

and we will show that L is parallel.

First, we shall prove that, if V is orthogonal to L , then $\nabla_{e_i}^\perp V$ is orthogonal to $JT\Sigma^2$ and to JH , where $\{e_1, e_2\}$ is a frame w.r.t. which we have $\langle \sigma(e_1, e_2), V \rangle = \langle \sigma(e_1, e_2), H \rangle = 0$. Indeed, we get

$$\begin{aligned} \langle (JH)^\perp, \nabla_{e_i}^\perp V \rangle &= \langle (JH)^\perp, \nabla_{e_i}^N V \rangle = -\langle \nabla_{e_i}^N (JH)^\perp, V \rangle \\ &= -\langle \nabla_{e_i}^N JH, V \rangle + \langle \nabla_{e_i}^N (JH)^\top, V \rangle \\ &= \langle JA_H e_i, V \rangle + \langle \sigma(e_i, (JH)^\top), V \rangle \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \langle (Je_j)^\perp, \nabla_{e_i}^\perp V \rangle &= -\langle \nabla_{e_i}^N (Je_j)^\perp, V \rangle \\ &= -\langle \nabla_{e_i}^N Je_j, V \rangle + \langle \nabla_{e_i}^N (Je_j)^\top, V \rangle \\ &= -\langle J\nabla_{e_i} e_j, V \rangle - \langle J\sigma(e_i, e_j), V \rangle + \langle \sigma(e_i, (Je_j)^\top), V \rangle \\ &= 0. \end{aligned}$$

Next, we shall prove that if a normal subbundle S is orthogonal to L , then so is $\nabla^\perp S$, i.e.

$$\langle \sigma(e_i, e_j), \nabla_{e_k}^\perp V \rangle = 0, \quad \langle J\sigma(e_i, e_j), \nabla_{e_k}^\perp V \rangle = 0 \quad \text{and} \quad \langle Je_i, \nabla_{e_k}^\perp V \rangle = 0$$

for any $V \in S$ and $i, j, k \in \{1, 2\}$. Since we have just proved the last property, it remains only to verify the first two of them.

We denote $A_{ijk} = \langle \nabla_{e_k}^\perp \sigma(e_i, e_j), V \rangle$ and, since σ is symmetric, we have $A_{ijk} = A_{jik}$. We also obtain $A_{ijk} = -\langle \sigma(e_i, e_j), \nabla_{e_k}^\perp V \rangle$, since V is orthogonal to L . We get

$$\begin{aligned} \langle (\nabla_{e_k}^\perp \sigma)(e_i, e_j), V \rangle &= \langle \nabla_{e_k}^\perp \sigma(e_i, e_j), V \rangle - \langle \sigma(\nabla_{e_k} e_i, e_j), V \rangle - \langle \sigma(e_i, \nabla_{e_k} e_j), V \rangle \\ &= \langle \nabla_{e_k}^\perp \sigma(e_i, e_j), V \rangle, \end{aligned}$$

and, from the Codazzi equation,

$$\begin{aligned}
\langle (\nabla_{e_k}^\perp \sigma)(e_i, e_j), V \rangle &= \langle (\nabla_{e_i}^\perp \sigma)(e_k, e_j) + (R^N(e_k, e_i)e_j)^\perp, V \rangle \\
&= \langle (\nabla_{e_j}^\perp \sigma)(e_k, e_i) + (R^N(e_k, e_j)e_i)^\perp, V \rangle \\
&= \langle (\nabla_{e_i}^\perp \sigma)(e_k, e_j), V \rangle = \langle (\nabla_{e_j}^\perp \sigma)(e_k, e_i), V \rangle.
\end{aligned}$$

We have just proved that $A_{ijk} = A_{kji} = A_{ikj}$.

Next, since $\nabla_{e_k}^\perp V$ is orthogonal to $JT\Sigma^2$ and to JH , it follows that the frame field $\{e_1, e_2\}$ diagonalizes $A_{\nabla_{e_k}^\perp V}$ and we get

$$A_{ijk} = -\langle \sigma(e_i, e_j), \nabla_{e_k}^\perp V \rangle = -\langle e_i, A_{\nabla_{e_k}^\perp V} e_j \rangle = 0$$

for any $i \neq j$. Hence, we have obtained that $A_{ijk} = 0$ if two indices are different from each other.

Finally, we only have to prove that $A_{iii} = 0$. Indeed, we have

$$\begin{aligned}
A_{iii} &= -\langle \sigma(e_i, e_i), \nabla_{e_i}^\perp V \rangle = -\langle 2H, \nabla_{e_i}^\perp V \rangle + \langle \sigma(e_j, e_j), \nabla_{e_i}^\perp V \rangle \\
&= \langle 2\nabla_{e_i}^\perp H, V \rangle - A_{jji} = 0.
\end{aligned}$$

It is easy to see that if V is orthogonal to L , then JV is normal and orthogonal to L . It follows that

$$\begin{aligned}
\langle (J\sigma(e_i, e_j))^\perp, \nabla_{e_k}^\perp V \rangle &= -\langle \nabla_{e_k}^N (J\sigma(e_i, e_j))^\perp, V \rangle \\
&= -\langle \nabla_{e_k}^N J\sigma(e_i, e_j), V \rangle + \langle \nabla_{e_k}^N (J\sigma(e_i, e_j))^\top, V \rangle \\
&= \langle JA_{\sigma(e_i, e_j)} e_k, V \rangle - \langle J\nabla_{e_k}^\perp \sigma(e_i, e_j), V \rangle \\
&\quad + \langle \sigma(e_k, (J\sigma(e_i, e_j))^\top), V \rangle \\
&= \langle \nabla_{e_k}^\perp \sigma(e_i, e_j), JV \rangle \\
&= 0.
\end{aligned}$$

Thus, we come to the conclusion that the subbundle L is parallel. \square

In the case when H is umbilical we can use the quadratic form Q to prove the following

Lemma 5.5. *Let Σ^2 be an immersed surface in a complex space form $N^n(\rho)$, $\rho \neq 0$, with nonzero parallel mean curvature vector H . If H is an umbilical direction everywhere, then Σ^2 is a totally real pseudo-umbilical surface of N .*

Proof. Since H is umbilical it follows that $\langle \sigma(Z, Z), H \rangle = 0$, which implies that Σ^2 is pseudo-umbilical and that $Q(Z, Z) = 3\rho \langle JZ, H \rangle^2$.

Next, as the $(2, 0)$ -part of Q is holomorphic, we have $\bar{Z}(Q(Z, Z)) = 0$ and further

$$0 = \bar{Z}(\langle JZ, H \rangle^2) = -2|H|^2 \langle JZ, H \rangle \langle JZ, \bar{Z} \rangle,$$

as we have seen in a previous section. Hence, $\langle JZ, \bar{Z} \rangle = 0$ or $\langle JZ, H \rangle = 0$. Assume that the set of zeroes of $\langle JZ, \bar{Z} \rangle = 0$ is not the entire Σ^2 . Then, by analyticity, it is a closed set without interior points and its complement is an open dense set in

Σ^2 . In this last set we have $\langle JZ, H \rangle = 0$ and then, since H is parallel and Σ^2 is pseudo-umbilical,

$$\begin{aligned} 0 = \bar{Z}(\langle JZ, H \rangle) &= \langle J\nabla_{\bar{Z}}^N Z, H \rangle + \langle JZ, \nabla_{\bar{Z}}^N H \rangle \\ &= -\langle \bar{Z}, Z \rangle \langle JH, H \rangle - \langle JZ, A_H \bar{Z} \rangle \\ &= -|H|^2 \langle JZ, \bar{Z} \rangle, \end{aligned}$$

which means that Σ^2 is also totally real. \square

Remark 5.6. Some kind of a converse result was obtained by B.-Y. Chen and K. Ogiue since they proved in [7] that if a unit normal vector field to a 2-sphere, immersed in a complex space form as a totally real surface, is parallel and isoperimetric, then it is umbilical.

Remark 5.7. In [19] N. Sato proved that, if M is a pseudo-umbilical submanifold of a complex projective space $\mathbb{C}P^n(\rho)$, with nonzero parallel mean curvature vector field, then it is a totally real submanifold. Moreover, the mean curvature vector field H is orthogonal to JTM . Therefore, if M is a surface, it follows that the $(2,0)$ -part of Q vanishes on M .

Remark 5.8. In order to show that only the two situations exposed in Lemma 5.4 and Lemma 5.5 can occur, we shall use an argument similar to that in Remark 5 in [4]. Thus, since the map $p \in \Sigma^2 \rightarrow (A_H - \mu I)(p)$, where μ is a constant, is analytic, it follows that if H is an umbilical direction, then this either holds on Σ^2 or only for a closed set without interior points. In this second case H is not an umbilical direction in an open dense set, and then Lemma 5.4 holds on this set. By continuity it holds on Σ^2 .

By using Lemma 5.4 and Lemma 5.5 we can state

Proposition 5.9. *Either H is everywhere an umbilical direction, and Σ^2 is a totally real pseudo-umbilical surface of N , or H is nowhere an umbilical direction, and there exists a subbundle of the normal bundle that is parallel, contains the image of the second fundamental form and its dimension is less or equal to 8.*

Now, from Proposition 5.9 and a result of J. H. Eschenburg and R. Tribuzy (Theorem 2 in [11]), it follows

Theorem 5.10. *Let Σ^2 be an isometrically immersed surface in a complex space form $N^n(\rho)$, $n \geq 3$, $\rho \neq 0$, with nonzero parallel mean curvature vector. Then, one of the following holds:*

- (1) Σ^2 is a totally real pseudo-umbilical surface of $N^n(\rho)$, or
- (2) Σ^2 is not pseudo-umbilical and it lies in a complex space form $N^r(\rho)$, where $r \leq 5$.

Remark 5.11. The case when $\rho = 0$ is solved by Theorem 4 in [21].

Remark 5.12. We have seen (Remark 5.6) that if Σ^2 is a totally real 2-sphere then it is pseudo-umbilical and therefore the second case of the previous Theorem cannot occur for such surfaces.

6. 2-SPHERES WITH CONSTANT KÄHLER ANGLE IN COMPLEX SPACE FORMS

This section is devoted to the study of immersed surfaces Σ^2 in a complex space form $N^n(\rho)$, $n \geq 3$, $\rho \neq 0$, with nonzero non-umbilical parallel mean curvature

vector H and constant Kähler angle, on which the $(2,0)$ -part of Q vanishes. We shall compute the Laplacian of the function $|A_H|^2$ for such a surface and show that there are no 2-spheres with these properties.

Let $\{e_1, e_2\}$ be an orthonormal frame on Σ^2 such that $H \perp Je_1$. The fact that the $(2,0)$ -part of the quadratic form Q vanishes can be written as

$$(6.1) \quad \begin{cases} 8|H|^2 \langle \sigma(e_1, e_1) - \sigma(e_2, e_2), H \rangle = -3\rho(\langle Je_1, H \rangle^2 - \langle Je_2, H \rangle^2) \\ 8|H|^2 \langle \sigma(e_1, e_2), H \rangle = 3\rho \langle Je_1, H \rangle \langle Je_2, H \rangle, \end{cases}$$

and, from the second equation, we see that $\langle \sigma(e_1, e_2), H \rangle = 0$. It follows that the frame $\{e_1, e_2\}$ diagonalizes simultaneously A_H and A_V , for all normal vectors V as in Corollary 5.3, since we are in the second case of Theorem 5.10.

Next, since Σ^2 is not holomorphic or anti-holomorphic, we have $\cos \theta \neq \pm 1$ on an open dense set and we can consider again the normal vectors

$$e_3 = -\cot \theta e_1 - \frac{1}{\sin \theta} Je_2 \quad \text{and} \quad e_4 = \frac{1}{\sin \theta} Je_1 - \cot \theta e_2$$

and obtain an orthonormal frame $\{e_1, e_2, e_3, e_4\}$ in $\text{span}\{e_1, e_2, Je_1, Je_2\}$, where θ is the Kähler angle on Σ^2 .

It is easy to see that if $H \perp JT\Sigma^2$ it results that the surface is pseudo-umbilical, which is a contradiction.

On the other hand, if we assume that $H \in \text{span}\{e_3, e_4\}$ it follows $H = \pm|H|e_3$, since $Je_1 \perp H$, and then e_3 is parallel. Also, since all normal vectors but e_4 verify conditions in Corollary 5.3 we have $\sigma(e_1, e_2) \parallel e_4$. By using these facts and the expression of e_3 we obtain that $\sigma(e_i, e_j) \in \text{span}\{e_3, e_4\}$ for $i, j \in \{1, 2\}$, and then $\dim L = 2$, where L is the subbundle in Lemma 5.4. Therefore, again by the meaning of Theorem 2 in [11], we get that Σ^2 lies in a complex space form $N^2(\rho)$, which case was studied earlier in this paper.

Consequently, in the following, we shall assume that $H \notin \text{span}\{e_3, e_4\}$, and, as we also know that H is not orthogonal to $JT\Sigma^2$, it results that the mean curvature vector can be written as

$$H = |H|(\cos \beta e_3 + \sin \beta e_5)$$

where β is a real-valued function defined locally on Σ^2 and e_5 is a unit normal vector field such that $e_5 \perp JT\Sigma^2$. We consider the orthonormal frame field

$$\{e_1, e_2, e_3, e_4, e_5, e_6 = Je_5, \dots, e_{2n-1}, e_{2n} = Je_{2n-1}\}$$

on N and its dual frame $\{\theta_i\}_{i=1}^{2n}$. These are well defined at the points of Σ^2 where $\sin(2\beta) \neq 0$, which, due to our assumptions, form an open dense set in Σ^2 . The structure equations of the surface are

$$d\phi = -i\theta_{12} \wedge \phi \quad \text{and} \quad d\theta_{12} = -\frac{i}{2}K\phi \wedge \bar{\phi},$$

where $\phi = \theta_1 + i\theta_2$, the real 1-form θ_{12} is the connection form of the Riemannian metric on Σ^2 and K is the Gaussian curvature.

A result of T. Ogata in [17], together with $H \perp e_i$ for any $i \geq 4$, $i \neq 5$, imply that, w.r.t. the above orthonormal frame, the components of the second fundamental form are

$$\begin{aligned} \sigma^3 &= \begin{pmatrix} |H| \cos \beta - \Re(\bar{a} + c) & -\Im(\bar{a} + c) \\ -\Im(\bar{a} + c) & |H| \cos \beta + \Re(\bar{a} + c) \end{pmatrix}, \quad \sigma^4 = \begin{pmatrix} \Im(\bar{a} - c) & -\Re(\bar{a} - c) \\ -\Re(\bar{a} - c) & -\Im(\bar{a} - c) \end{pmatrix} \\ \sigma^5 &= \begin{pmatrix} |H| \sin \beta - \Re(\bar{a}_3 + c_3) & -\Im(\bar{a}_3 + c_3) \\ -\Im(\bar{a}_3 + c_3) & |H| \sin \beta + \Re(\bar{a}_3 + c_3) \end{pmatrix} \end{aligned}$$

$$\sigma^6 = \begin{pmatrix} \Im(\bar{a}_3 - c_3) & -\Re(\bar{a}_3 - c_3) \\ -\Re(\bar{a}_3 - c_3) & -\Im(\bar{a}_3 - c_3) \end{pmatrix}$$

$$\sigma^{2\alpha-1} = \begin{pmatrix} -\Re(\bar{a}_\alpha + c_\alpha) & -\Im(\bar{a}_\alpha + c_\alpha) \\ -\Im(\bar{a}_\alpha + c_\alpha) & \Re(\bar{a}_\alpha + c_\alpha) \end{pmatrix}, \quad \sigma^{2\alpha} = \begin{pmatrix} \Im(\bar{a}_\alpha - c_\alpha) & -\Re(\bar{a}_\alpha - c_\alpha) \\ -\Re(\bar{a}_\alpha - c_\alpha) & -\Im(\bar{a}_\alpha - c_\alpha) \end{pmatrix}$$

where a, c, a_α, c_α , with $\alpha \in \{3, \dots, n\}$, are complex-valued functions defined locally on the surface Σ^2 . We note that, since $\sigma(e_1, e_2) \perp H$ and $\sigma(e_1, e_2) \perp e_5$, it follows $\sigma(e_1, e_2) \perp e_3$. Moreover, since $\sigma(e_1, e_2) \perp e_i$ for any $i \in \{1, \dots, 2n\} \setminus \{4, 6\}$, we have $\bar{a} + c \in \mathbb{R}$, $\bar{a}_3 + c_3 \in \mathbb{R}$ and $a_\alpha = c_\alpha$ for any $\alpha \geq 4$.

In the same paper [17], amongst others, the author computed the differential of the Kähler angle function θ for a minimal surface. In the same way, this time for our surface, we get

$$d\theta = \left(a - \frac{|H|}{2} \cos \beta\right) \phi + \left(\bar{a} - \frac{|H|}{2} \cos \beta\right) \bar{\phi}.$$

The next step is to determine the connection form θ_{12} and the differential of the function β , by using the property of H being parallel. We have

$$(6.2) \quad \nabla_{e_i}^\perp H = (-\sin \beta e_3 + \cos \beta e_5) d\beta(e_i) + \cos \beta \nabla_{e_i}^\perp e_3 + \sin \beta \nabla_{e_i}^\perp e_5 = 0$$

for $i \in \{1, 2\}$, and then

$$\cos \beta \langle \nabla_{e_i}^N e_3, e_4 \rangle + \sin \beta \langle \nabla_{e_i}^N e_3, e_4 \rangle = 0, \quad i \in \{1, 2\}$$

from where, by using the expressions of e_3 in the first term, of e_4 in the second one and of the second fundamental form of Σ^2 , we get

$$\theta_{12}(e_1) = \cot \theta \Im(\bar{a} - c) - \frac{\tan \beta}{\sin \theta} \Im(\bar{a}_3 - c_3)$$

$$\theta_{12}(e_2) = -|H| \frac{\cot \theta}{\cos \beta} - 2 \cot \theta \Re a + \tan \beta \left(\tan \left(\frac{\theta}{2} \right) \Re a_3 - \cot \left(\frac{\theta}{2} \right) \Re c_3 \right)$$

and finally $\theta_{12} = f_1 \phi + \bar{f}_1 \bar{\phi}$, where

$$(6.3) \quad f_1 = \frac{i}{2} \left(|H| \frac{\cot \theta}{\cos \beta} + 2 \cot \theta a - \frac{\tan \beta}{\sin \theta} (a_3 - \bar{c}_3) + \cot \theta \tan \beta (a_3 + \bar{c}_3) \right).$$

Now, from equation (6.2), we also obtain

$$d\beta(e_i) + \langle \nabla_{e_i}^N e_3, e_5 \rangle = 0, \quad i \in \{1, 2\}$$

and then, replacing e_3 with its expression and also using the expression of the second fundamental form, we get

$$d\beta(e_1) = |H| \cot \theta \sin \beta + \tan \left(\frac{\theta}{2} \right) \Re a_3 - \cot \left(\frac{\theta}{2} \right) \Re c_3, \quad d\beta(e_2) = \frac{1}{\sin \theta} \Im(\bar{a}_3 - c_3).$$

Hence the differential of β is given by $d\beta = f_2 \phi + \bar{f}_2 \bar{\phi}$, where

$$(6.4) \quad f_2 = \frac{1}{2} \left(|H| \cot \theta \sin \beta + \frac{1}{\sin \theta} (a_3 - \bar{c}_3) - \cot \theta (a_3 + \bar{c}_3) \right).$$

We note that if the Kähler angle θ is constant, then $a = \bar{a} = \frac{|H|}{2} \cos \beta$, and, from (6.3), it results

$$(6.5) \quad f_1 = \frac{i}{2} \left\{ |H| \cot \theta \left(\cos \beta + \frac{1}{\cos \beta} \right) - \frac{\tan \beta}{\sin \theta} (a_3 - \bar{c}_3) + \cot \theta \tan \beta (a_3 + \bar{c}_3) \right\}.$$

Let us now return to the first equation of (6.1), which can be rewritten as

$$\mu_1 - \mu_2 = \frac{3}{8} \rho \sin^2 \theta \cos^2 \beta,$$

where $A_H e_i = \mu_i e_i$. Since $\mu_1 + \mu_2 = 2|H|^2$ we have $\mu_1 = |H|^2 + \frac{3}{16}\rho \sin^2 \theta \cos^2 \beta$ and $\mu_2 = |H|^2 - \frac{3}{16}\rho \sin^2 \theta \cos^2 \beta$. Thus

$$(6.6) \quad |A_H|^2 = \mu_1^2 + \mu_2^2 = 2|H|^4 + \frac{9}{128}\rho^2 \sin^4 \theta \cos^4 \beta.$$

In the following, we shall assume that the Kähler angle of the surface Σ^2 is constant and then the Laplacian of $|A_H|^2$ is given by

$$\Delta |A_H|^2 = \frac{9}{128}\rho^2 \sin^4 \theta \Delta(\cos^4 \beta).$$

In order to compute the Laplacian of $\cos^4 \beta$ we need the following formula, obtained by using (6.4) and (6.5),

$$\begin{aligned} d(\cos^4 \beta) &= -4 \sin \beta \cos^3 \beta d\beta = -4 \sin \beta \cos^3 \beta (f_2 \phi + \bar{f}_2 \bar{\phi}) \\ &= -4 \cos^4 \beta \left\{ \left(i f_1 + |H| \frac{\cot \theta}{\cos \beta} \right) \phi + \left(-i \bar{f}_1 + |H| \frac{\cot \theta}{\cos \beta} \right) \bar{\phi} \right\}. \end{aligned}$$

We also have $dd^c(\cos^4 \beta) = \frac{i}{2}(\Delta(\cos^4 \beta))\phi \wedge \bar{\phi}$ and

$$d^c(\cos^4 \beta) = -4i \cos^4 \beta \left\{ \left(-i \bar{f}_1 + |H| \frac{\cot \theta}{\cos \beta} \right) \bar{\phi} - \left(i f_1 + |H| \frac{\cot \theta}{\cos \beta} \right) \phi \right\}.$$

After a straightforward computation, we get

$$\Delta(\cos^4 \beta) = 4 \cos^4 \beta \left(K + 4|f_1|^2 + 12 \left| i f_1 + |H| \frac{\cot \theta}{\cos \beta} \right|^2 \right)$$

and then

$$\Delta |A_H|^2 = \frac{9}{32}\rho^2 \sin^4 \theta \cos^4 \beta \left(K + 4|f_1|^2 + 12 \left| i f_1 + |H| \frac{\cot \theta}{\cos \beta} \right|^2 \right).$$

Assume now that the surface Σ^2 is complete and it has nonnegative Gaussian curvature. It follows, from a result of A. Huber in [13], that Σ^2 is parabolic. Then, from the above formula, we get that $|A_H|^2$ is a subharmonic function, and, since $|A_H|^2$ is bounded (due to (6.6)), it results $K = 0$, which, together with the Gauss-Bonnet Theorem, lead to the following non-existence result.

Theorem 6.1. *There are no 2-spheres with nonzero non-umbilical parallel curvature vector and constant Kähler angle in a non-flat complex space form.*

REFERENCES

- [1] U. Abresch and H. Rosenberg, *A Hopf differential for constant mean curvature surfaces in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$* , Acta Math. 193(2004), 141–174.
- [2] U. Abresch and H. Rosenberg, *Generalized Hopf differentials*, Mat. Contemp. 28(2005), 1–28.
- [3] H. Alencar, M. do Carmo and R. Tribuzy, *A theorem of Hopf and the Cauchy-Riemann inequality*, Comm. Anal. Geom. 15(2007), 283–298.
- [4] H. Alencar, M. do Carmo and R. Tribuzy, *A Hopf Theorem for ambient spaces of dimensions higher than three*, J. Differential Geometry 84(2010), 1–17.
- [5] J. Bolton, G.R. Jensen, M. Rigoli and L.M. Woodward, *On conformal minimal immersions of \mathbb{S}^2 into \mathbb{CP}^n* , Math. Ann. 279(1988), 599–620.
- [6] B.-Y. Chen, *Special slant surfaces and a basic inequality*, Results Math. 33(1998), 65–78.
- [7] B.-Y. Chen and K. Ogiue, *On totally real submanifolds*, Trans. Am. Math. Soc. 193(1974), 257–266.
- [8] S.-S. Chern, *On surfaces of constant mean curvature in a three-dimensional space of constant curvature*, Geometric dynamics (Rio de Janeiro, 1981), Lecture Notes in Math. 1007, Springer, Berlin, 1983, 104–108.
- [9] S.-S. Chern and J. Wolfson, *Minimal surfaces by moving frames*, Amer. J. Math 105(1983), 59–83.

- [10] J.H. Eschenburg, I.V. Guadalupe and R. Tribuzy, *The fundamental equations of minimal surfaces in \mathbb{CP}^2* , Math. Ann. 270(1985), 571–598.
- [11] J.H. Eschenburg and R. Tribuzy, *Existence and uniqueness of maps into affine homogeneous spaces*, Rend. Sem. Mat. Univ. Padova 89(1993), 11–18.
- [12] S. Hirakawa, *Constant Gaussian curvature surfaces with parallel mean curvature vector in two-dimensional complex space forms*, Geom. Dedicata 118(2006), 229–244.
- [13] A. Huber, *On subharmonic functions and differential geometry in the large*, Comm. Math. Helv. 32(1957), 13–71.
- [14] H. Hopf, *Differential geometry in the large*, Lecture Notes in Math. 1000, Springer-Verlag, 1983.
- [15] K. Kenmotsu and D. Zhou, *The classification of the surfaces with parallel mean curvature vector in two-dimensional complex space forms*, Amer. J. Math. 122(2000), 295–317.
- [16] T. Ogata, *Surfaces with parallel mean curvature vector in $P^2(C)$* , Kodai Math. J. 18(1995), 397–407.
- [17] T. Ogata, *Curvature pinching theorem for minimal surfaces with constant Kähler angle in complex projective spaces*, Tôhoku Math. J. 43(1991), 361–374.
- [18] Y. Ohnita, *Minimal surfaces with constant curvature and Kähler angle in complex space forms*, Tsukuba J. Math. 13(1989), 191–207.
- [19] N. Sato, *Totally real submanifolds of a complex space form with nonzero parallel mean curvature vector*, Yokohama Math. J. 44(1997), 1–4.
- [20] F. Torralbo and F. Urbano, *Surfaces with parallel mean curvature vector in $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{H}^2 \times \mathbb{H}^2$* , Trans. Am. Math. Soc., to appear.
- [21] S.-T. Yau, *Submanifolds with constant mean curvature*, Amer. J. Math. 96(1974), 346–366.

DEPARTMENT OF MATHEMATICS, "GH. ASACHI" TECHNICAL UNIVERSITY OF IASI, BD. CAROL I NO. 11, 700506 IASI, ROMANIA

E-mail address: dfetcu@math.tuiasi.ro

Current address: IMPA, Estrada Dona Castorina 110, 22460-320 Rio de Janeiro, Brazil

E-mail address: dorel@impa.br